Functional Solution Scheme for Relativistic Strong-coupling Theory III.

GERHARD HEBER, ADOLF KÜHNEL *

Theoretisch-Physikalisches Institut der Universität Leipzig

and H. J. KAISER

Arbeitsstelle für Hochenergiephysik der DAdW, Zeuthen bei Berlin

(Z. Naturforschg. **20 a**, 498–503 [1965]; received 14 December 1964)

The graph scheme developed in [1] is further investigated. The structure of the coefficients α_k is determined in terms of α_0 and the projections of the splitting graphs. This allows a formal summation of the k-series. The first few terms of the strong coupling series for the lowest mass eigenvalue are computed for the $\lambda\varphi^4$ theory. The leading term for large λ is complex. This restricts physically acceptable solutions to λ below some critical value.

By including a bare mass m in the Lagrangian, the α_k become dependent on it. Consideration of one α_k e. g. $\alpha_0(m, \lambda)$ should then be sufficient for the determination of physical mass eigenvalues.

This paper is a further development of the functional solution scheme for a relativistic strong coupling theory proposed in [1]. As it is done in [1], we deal with a real scalar field, self-coupled by a polynomial interaction Lagrangian about the special structure of which we need no further assumption. In [1] a power series with respect to p^2 has been found for the FOURIER-transform of the 2-point-function. We are able to sum this power series formally by means of an adequate representation of the coefficients in this series. In this way, we find an explicite expression for the situation of the pole nearest to the origin in the complex p^2-plane, valid for big values of the coupling constant. The poles at larger distances from the origin cannot be determined so easyly but we can give an approximate expression for the second pole (and in principle also for the higher ones). All these expressions are series with respect to decreasing powers of the coupling constant.

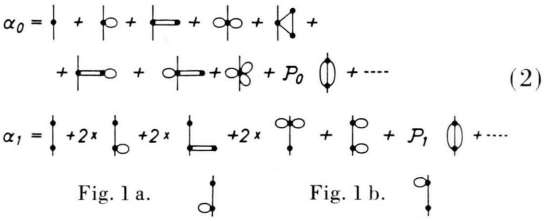
In particular we apply our results to a real scalar field model with the self-interaction-Lagrangian $\lambda\varphi^4$. Then we find for big λ a complex value for the nearest pole in the p^2-plane, the distance from the origin being proportional to $\lambda^{1/2}$. The higher poles (in complex p^2-plane, too) are given by decreasing power series starting with some higher order in $\lambda^{1/2}$ and progressing with $\lambda^{-1/2}$. – The possible physical meaning of these results is discussed.

1. An Useful Connection between the α_k

In [1] we have established that the FOURIER-Transform $T_2(p)$ of the 2-point-function can be written in the form

$$T_2(p) = -i\sum_{k=0}^{\infty}\alpha_k(p^2)^k. \tag{1}$$

The α_k are by their definition infinite series with respect to negative powers of the coupling constant [2]. A very useful tool to write down the α_k is the graphical representation given in [1], e. g. we can write [3].



(2)

Fig. 1 a. Fig. 1 b.

* Present adress: Joint Institute for Nuclear Research, Dubna/UdSSR.

[1] G. HEBER and H. J. KAISER, Z. Naturforschg. **19 a**, 828 [1964].

[2] The particular exponent of λ in the terms of α_k depends, of course, on the special structure of the interaction LAGRANGian. We call the power of λ in the first term of α_0 η and say thereafter, we have series with respect to increasing integer powers of η.

[3] In equ. (2) the numerical coefficients are the "multiplicities" of the graphs. They account for the numerical equality of e. g. the graphs of Fig. 1 a and 1 b and are not to be confused with the weight factors (I, Appendix C). The symbols $\mathcal{P}_0$ and $\mathcal{P}_1$ mean the "projections" of $(\Box\,\delta(x))^3$ on $\delta(x)$ and $\Box\,\delta(x)$ resp. (see also Appendix A).

Let us assume the α_k in eqs. (1), (2) already „renormalized", i. e. they must not contain only longer the quantity ε (the limit of the volume of a primitive cell in lattice-space). As it is pointed out in [1], this renormalization is possible in those cases, which are renormalizable in the framework of perturbation theory [4]. The numerical value of such single graphs appearing in (2) can be found in App. A.

The problem to be solved is to sum the double infinite series (1). Some insight can be gained by use of the following representation for the α_k:

$$\left.\begin{aligned} &\alpha_0 \\ &\alpha_1 = \alpha_0^2 + g_1 , \\ &\alpha_2 = \alpha_0^3 + 2\,\alpha_0\, g_1 + g_2 , \\ &\alpha_3 = \alpha_0^4 + 3\,\alpha_0^2\, g_1 + 2\,\alpha_0\, g_2 + g_1^2 + g_3 , \end{aligned}\right\} \qquad (3)$$

and so on.

Formulae (3) need the following explanation: Multiplication of two (or more) graphs means their connection at their free ends, e. g. as in Fig. 2.

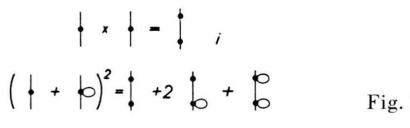


Fig. 2.

g_1 contains the contributions to α_1 which come from the splitting graphs and are not contained in α_0. In the graphical representation we have Fig. 3.

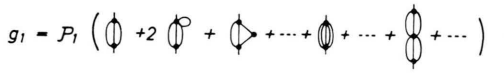


Fig. 3.

g_1 is a power-series with respect to η, too, starting with η^4. Multiplication is shown here in Fig. 4.

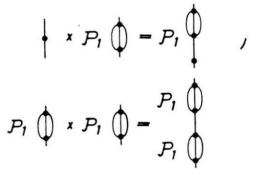


Fig. 4.

The product of two terms contained in g_1 appears first in α_3 but not in α_2. Further we have Fig. 5. g_2 starts with η^7.

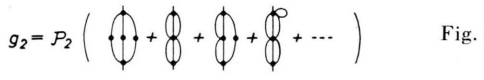


Fig. 5.

[4] In superrenormalizable theories as well as in nonrenormalizable ones an ε-dependence must be dealt with.

One can check easily the representation (3) of the α_k noting that the numbers n and k (see Appendix A) appear only as exponents and that the multiplications in (2) are just the multinomial-coefficients. These multiplicities are the numbers of possibilities to construct different graphs from different contributions to α_0. In (3) there again appear certain multiplicities which have a quite similar meaning, namely: they are the numbers of the possible different sequences of the factors α_0, g_1, g_2 in a product $\alpha_0^{\,i}\, g_1^{\,j}\, g_2^{\,k} \ldots$. Therefore, they are again multinomial-coefficients times the permutations of the different factors. We get for the general expression [5]

$$\alpha_k = \sum_{l_i} \frac{(k - \sum_{j=0}^{r} j\, l_j + 1)!}{l_0!\, l_1! \ldots l_r!}\, \alpha_0^{l_0}\, g_1^{l_1}\, g_2^{l_2} \ldots g_r^{l_r}. \qquad (4)$$

The sum appearing in the factorial is to be performed in such a way that the condition

$$\sum_{j=0}^{r} (j+1)\, l_j = k+1 \qquad (5)$$

is fulfilled. With this representation for α_k we are able to sum this series (1) over k.

2. Summation

We have to sum

$$\begin{aligned} T_2(p) &= -i \sum_{k=0}^{\infty} \alpha_k (p^2)^k \\ &= -i \sum_{k=0}^{\infty} \sum_{l_i} \frac{(k - \sum_{j=0}^{r} j\, l_j + 1)!}{\prod_{j=0}^{r} l_j!}\, \alpha_0^{l_0}\, g_1^{l_1}\, g_2^{l_2} \ldots g_r^{l_r}\, p^{2k} \end{aligned} \qquad (6)$$

with the restriction (5).

Because of (5) we have

$$\begin{aligned} &\alpha_0^{l_0}\, g_1^{l_1}\, g_2^{l_2} \ldots g_r^{l_r}\, p^{2k} \\ &\quad = (\alpha_0\, p^2)^{l_0}\, (g_1\, p^4)^{l_1}\, (g_2\, p^6)^{l_2} \ldots (g_r\, p^{2(r+1)})^{l_r}\, p^{-2}. \end{aligned}$$

Now we shall reorder the series (5) so that we gather no longer the terms for a fixed k but for a fixed value of the number

$$N = k - \sum_{j=0}^{r} j\, l_j + 1 . \qquad (7)$$

Then we have by (5) $l_0 = N - \sum_{j=1}^{r} l_j$ and the sum in

[5] The numerator of the weight factor is the number of permutations of all factors, the denominator the number of permutations of identical factors.

(6) starts as a sum over N with $N=1$ which is the value following from (7) for $k=0$. In this way we transform (6) into

$$T_2(p) = -i\,p^{-2}\sum_{N=1}^{\infty}\sum_{l_i}\frac{N!}{\prod_{j=0}^{r} l_j!}\,(\alpha_0\,p^2)^{l_0}\,(g_1\,p^4)^{l_1}\ldots\,(g_r\,p^{2(r+1)})^{l_r} = -i\,p^{-2}\sum_{N=1}^{\infty}(\alpha_0\,p^2+g_1\,p^4+g_2\,p^6+\ldots)^N . \tag{8}$$

The second line of (8) follows with the help of the polynomial formula. Now, we substitute N by $N'+1$ and call N' again N. The new summation index runs from 0 to ∞. We get:

$$T_2(p) = -i\,p^{-2}\sum_{N=0}^{\infty}(\alpha_0\,p^2+g_1\,p^4+\ldots)^{N+1} = -i\,\frac{\alpha_0+\sum_{i=1}^{\infty}g_i\,p^{2i}}{1-p^2(\alpha_0+\sum_{j=1}^{\infty}g_j\,p^{2j})} \tag{9}$$

after summing up the resulting geometrical series. Expression (9) is a somewhat more closed expression for $T_2(p)$ than the power series (1). However, it contains still the infinite series $\Sigma\, g_i\,p^{2i}$. Further, we must remember that α_0 and the g_i are infinite series with respect to decreasing powers of some root of the coupling constant λ. But nevertheless, we have got an expression which is more adequate for the determination of the poles of $T_2(p)$ in which we are interested. – The residues of the poles can, of course, be calculated too. That one of the lowest pole is $-i$ (in the first four orders).

3. The Lowest Poles

The poles of $T_2(p)$ should correspond to mass-eigenvalues of the field under consideration. To find the poles of $T_2(p)$ we have to determine the zeros of the denominator:

$$1-p^2\,\alpha_0-p^2\sum_{i=1}^{\infty}g_i\,p^{2i}=0\,. \tag{10}$$

For the determination of the zeros we need the exact analytical behaviour of the denominator, but we have only a power series for it which is an inadequate tool for the detection of zeros. We shall do some neglections in (10) to get informations about the poles of $T_2(p)$, for we are not able to sum the series $\sum_{i=1}^{\infty} g_i\,p^{2i}$ exactly.

If we neglect the whole series $\sum_{i=1}^{\infty} g_i\,p^{2i}$ in (10), we get a pole for $p^2=K_0^{\,2}=1/\alpha_0\propto\eta^{-1}$ and no other poles.

Now we insert the found approximate value for $K_0^{\,2}$ into equ. (10) and find this equation to be fulfilled in the lowest two orders of η.

Going further and including the term $g_1\,p^4$ one gets a quadratic equation, one root of which is again α_0^{-1}, corrected by higher order terms [starting in order η, see equ. (12 a)]. The other root is $\propto\eta^{-3}$. If one inserts this second root into equ. (10) one finds that all terms of the infinite series are of the same order of magnitude with respect to η. Therefore it is no good approximation for the second pole to include only $g_1\,p^4$, for the corrections resulting from the other terms are of the same order η^{-3}.

Another way to treat equ. (10) consists in summing the lowest contributions (with respect to η) to the g_i: Doing this (see Appendix B) we get the quadratic equation

$$p^4\left(\frac{f_4^{\,2}}{6}-\alpha_0\,s\right)+p^2(s+16\,\alpha_0)-16=0 \tag{11}$$

with $\qquad s=-i(f_2^{\,3}+\frac{3}{2}f_2\,f_4+\frac{1}{6}f_6)\,.$

f_n is identical with $f_0^{(n)}$ in [1]. The roots of (11) are (in the lowest two orders of η)[6]:

$$K_0^{\,2}=\frac{1}{\alpha_0}\,,\qquad K_1^{\,2}=-\frac{16\,\alpha_0}{f_4^{\,2}/6-\alpha_0\,s}\,. \tag{12), (13}$$

It is not hard to give the first correction to the expression (12) for the first pole by going back to equ. (10) as indicated above. Then one gets in the next order:

$$K_0^{\,2}=\frac{1}{\alpha_0}\left(1-\frac{g_1}{\alpha_0^{\,2}}\right). \tag{12 a}$$

Corrections to (13) cannot be obtained so simply. One has to sum up for this the series $\sum_{i=1}^{\infty} g_i\,p^{2i}$ in a

[6] Remark that the expressions (12) and (12 a) for $K_0^{\,2}$ could also (in the valid order of η) be obtained from the determination of the radius of convergence of the Taylor-series (1). This method was proposed in [1]. It seems, however, not to be very simple to go in this way to the other poles (see also Appendix C).

better approximation. This can be done for the next two orders (with respect to η) with a similar technique as indicated in Appendix B. By summing up all graphs of the form Fig. 6, and the equivalent ones for g_1 (similar for the other g_i) one gets corrections

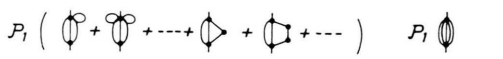


Fig. 6.

Fig. 6 a.

to K_1^2 and K_0^2 but no third pole. An equation of higher order in p^2 than 2 instead of (11) appears however first by taking into account graphs of the form Fig. 6 a in g_1 (and the corresponding in the other g_i). So the third pole does originate essentially from these five-fold splitting graphs, but of course these graphs produce also further corrections to (12 a) and (13). We do not wish to go into details here but we mention that according to (13) the power series for K_1^2 begins with η^{-3}. That for K_2^2 seems to begin with η^{-5}.

4. Application to $\lambda\varphi^4$ Model

We have applied our scheme to the special case of $L = \frac{1}{2}\varphi\Box\varphi - \lambda\varphi^4$. In this case we get

$$f_2 = \frac{0{,}338}{\sqrt{i\lambda}}, \quad f_4 = -\frac{0{,}0927}{i\lambda}, \quad f_6 = \frac{0{,}144}{\sqrt{i\lambda^3}}. \tag{14}$$

The explicit form of α_0 is according to (2):

$$\alpha_0 = i f_2 + \tfrac{1}{2} f_4 - \frac{3i}{4} f_4 f_2 - \frac{i}{8} f_6 + \ldots . \tag{15}$$

Using (14) and (15) we calculate K_0^2 from (12). In the lowest order, we find

$$K_0^2 = \frac{1}{i f_2} = \frac{i\sqrt{i}\sqrt{\lambda}}{0{,}338},$$

i. e. a complex value for the square of the lowest mass. This result, however, cannot be interpreted as an instable state (say resonance), for there are no particles with lower mass into which this state could decay. Therefore, we require the lowest pole to lie on the real positive p^2-axis. This is also to be postulated, as otherwise the two-point-function in x-space would for large x^2 have the wrong physical behaviour. It is evident that K_0^2 will be real only for certain values of λ. Calculating the value of λ for which K_0^2 lies on the real axis (for φ^4 model) we find $\lambda \approx 0{,}01$, if we use the first three terms in the expansion of K_0^2 according to (12 a). Of course, our method (series in decreasing powers of $\lambda^{1/2}$) is not a good one for such small λ. Not only the expression (12 a) becomes invalid for such small λ, it is also unknown whether K_0^2 is in this region still the lowest pole. So we can say only that for big λ the lowest pole does not lie on the real axis. If the spectrum contains the eigenvalue $K_0^2 = 0$ for some λ_c then all α_k become infinite there such that their power series in $\lambda^{-1/2}$ diverges for

$$\lambda \leqq \lambda_c .$$

This statement can be reversed: Unless, in the $\lambda\varphi^4$ theory, the strong coupling series converges throughout the halfplane $\Re\mathfrak{e}\,\lambda^{-1/2} > 0$ there must exist for some λ_c a zero eigenvalue.

Adding a bare mass m to the $\lambda\varphi^4$ theory we can by similar reasoning determine the position of the eigenvalues K_i^2 by the following procedure: Fix a point m and determine the largest λ for which the expansion of $\alpha_0(\lambda, m)$ as power series in $\lambda^{-1/2}$ diverges. Then for this λ one of the physical masses is $K_i = m$ (cf. Appendix C).

Higher poles may for special λ-values happen to lie on the real axis, but we have not investigated this topic further, as the lowest pole is the one most important for the asymptotic behaviour of the propagator for big x^2. – We remark that this unphysical behaviour of the propagator is a feature of the special model discussed in this section; it possibly is present, however, also in similar models [e. g. such with $L_W = -\lambda\varphi^M(x)$].

5. Discussion

What is the physical meaning of the special result derived in the last section? First we wish to stress the fact that judging from the first terms of the series K_0^2 becomes real only for small values of λ; this demonstrates that for big λ the original value of $K_0^2 = 1/(i f_2)$ is only very slowly shifted by the correction terms. That might mean that our formula (12 a) is indeed a good approximation for big λ. At least the next few correction terms are small for big λ.

Secondly we must remark that our calculation does not represent a rigorous proof of the unphysical behaviour of the propagator for big λ, as we are unable to sum up the series $\sum_i g_i p^{2i}$ exactly. We believe, however, that our result is correct for big λ and that this special feature of the model has something to do with lack of unitary for big λ. We mention

that SCHIFF [7] and MARX [8] have shown for the $\lambda\varphi^4$ model that the scattering amplitude is finite only after renormalizing the coupling constant such that the unrenormalized coupling constant λ must be zero. If this should be correct in that case also for big λ, the model has no physically acceptable solutions.

So it may even be that our method in this special case makes no sense, because of our assumption of big λ. But in models or systems which allow a big unrenormalized coupling-constant [9], our method should be a useful one. Finally we wish to remark that quite recently CAIANIELLO [11] put forward the supposition that the model, considered also by us, has GREEN's Functions being non-analytic at $\lambda = 0$. He supposes that there are branchpoints at $\lambda = 0$ and $\lambda = \infty$. He supposes further that these functions are analytic functions, however, of $\lambda^{-1/2}$. If this would be true, our expansion in the case $M = 4$ would be just the right one, as our expansion-parameter η is in that case in fact $\lambda^{-1/2}$. But CAIANIELLO does not yet have a complete proof [11].

Acknowledgement

It is a pleasure for us to thank the Direction and staff of the Joint Institute for Nuclear Research (Dubna) for the kind hospitality. Further we wish to thank Prof. NOVOSHILOW, Prof. TULUB, Dr. BRAUN (Leningrad) and many participants of the International Conference on High-Energy Physics (Aug. 1964, Dubna) for very stimulating discussions.

Numerical Value of a Graph

Appendix A

The numerical value of a graph is a product of five factors:

1. The factor $\dfrac{(-i)^{n/2-1}}{(n/2-1)!\,2^{n/2-1}}$ from the expansion (8) in [1], $2^{-n/2+1}$ arises from the fact that we use $L = -\frac{1}{2}\,\varphi \Box \varphi - \lambda\varphi^M$ instead of $L = \varphi \Box \varphi - \lambda\varphi^M$ in [1], but the same $G(i, j)$.

2. The weight factor $\dfrac{2^{n/2-1}(n/2-1)!}{2^s\, \Pi\, \varrho_i!\, \Pi\, j_i!}$ (see [1]), Appendix C, with substitution $g_i \to j_i$).

3. The corresponding product of f's, given in [1], Appendix A, which we call simply f.

4. The product of $\Box\,\delta(0)$, $\Box^2\,\delta(0)$, $(\Box\,\delta(0))^2, \ldots$ and the projections of the splitting graphs, corresponding to the given shape of the graph, which we call z.

5. A factor $(-1)^k$ from $\Box^k e^{-ipx} = (-p^2)^k e^{-ipx}$.

From all the product we exclude a factor i which is a common multiplier of all graphs, and write it in front of the whole expression for $T_2(p)$, and we get

$$\frac{(-i)^{n/2}\,(-1)^{n/2+1}\,(-1)^k}{2^s\, \Pi\, p_i!\, \Pi\, j_i!} \cdot f\, z$$

for the numerical value of a graph.

Remark, that ε has been removed in the case $M = 4$, $N = 4$ already by renormalization.

Regarding the factor z we give some examples. If one has only one simple loop, $\Box\,\delta(0)$ appears.

This can be evaluated with the help of formula (16 a) of [1] it yields $\Box\,\delta(0) = \varepsilon^{-3/2}$. Now ε is already extracted by renormalization, so we are left with $z = 1$ in that case.

Another case is $\Box^2\,\delta(0)$. Application of the same formula gives $z = 3/2$ for this case.

Further we give an example for the projection of a splitting-graph. There appears for example $(\Box\,\delta(x))^3$. which should be written as a linear combination:

$$(\Box\,\delta(x))^3 = a_1 \Box\,\delta(x) + a_0\,\delta(x)\ .$$

Higher derivatives cannot appear, as $\Box\,\delta(x)$ is not zero only at those points of lattice-space, where $x = 2\sqrt[4]{\varepsilon}$ or $x = 0$. [Formula (16) of [1] gives $\Box\,\delta(x) = 1/4$ for $x = 2\sqrt[4]{\varepsilon}$.] Now we must postulate that

$$(\Box\,\delta(x))^3 = a_1 \Box\,\delta(x) \quad \text{for} \quad x = 2\sqrt[4]{\varepsilon}\,.$$

It follows $a_1 = 1/4^2$. The determination of a_0 in

$$(\Box\,\delta(x))^3 = a_1 \Box\,\delta(x) + a_0\,\delta(x)$$

must be done similarly by comparision of the two sides of the equation for $x = 0$, a_1 being as above.

[7] L. I. SCHIFF, Proc. High-Energy-Physics Conf. at CERN, Geneva 1962, p. 690.

[8] G. MARX and G. KUTI, private communication.

[9] From unitary according to theorems of RUDERMANN et al. [10] limits can be derived for the renormalized coupling constant. But according to LEHMANN–KÄLLEN spectral representation, the renormalized coupling constant should be smaller than the non-renormalized one (at least in theories of the type of quantum electrodynamics).

[10] M. A. RUDERMANN, Phys. Rev. **127**, 312 [1962]. — S. D. DRELL, A. C. FINN, and A. C. HEARN, Rep. Intern. Conf. High-Energy-Physics, Dubna 1964.

[11] R. E. CAIANIELLO, private communication.

Appendix B: Summation of the lowest contributions to g_i

The lowest contribution to g_1 comes from the graph Fig. 7 and is $\frac{1}{16}\frac{1}{3!} f_4^2$. The factor $1/16$ results from the projection (see Appendix A); $1/3$ is the weight of the graph. The lowest order contributions to g_2 are shown in Fig. 8, which represent

Fig. 7.

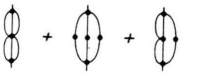


Fig. 8.

$$-\frac{1}{16^2}\left(\frac{1}{3!^2} f_4^2 f_6 + \frac{1}{3!} f_4^2 f_2^3 + \frac{1}{2!} f_4^3 f_2\right) = -\frac{i}{16^2}\frac{f_4^2}{3!}\left(\frac{1}{3!} f_6 + f_3^2 + \tfrac{3}{2} f_4 f_2\right) = \frac{f_4^2}{16^2 \cdot 3!} r ,$$

with

$$r = -i\left(\frac{1}{3!} f_6 + f_2^3 + \tfrac{3}{2} f_4 f_2\right) .$$

For g_3 we have in the considered approximation the graphs of Fig. 9, which represent

$$-\frac{1}{16^3} f_4^2 \left\{\frac{1}{3!^2} f_6^2 + \frac{1}{3!} f_2^6 + \left(\frac{1}{2!^3} + \frac{1}{2!^2}\right) f_4^2 f_2^2 + 2\frac{f_6 f_2^3}{3!^2} + 2\frac{f_6 f_4 f_2}{3!\, 2^2} + 2\frac{f_4 f_2^4}{2!^2}\right\} = \frac{1}{16^3}\frac{f_4^2}{3!} r^2 .$$

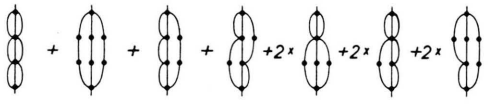


Fig. 9.

Further by induction, one finds

$$g_i = \frac{1}{16^i} \cdot \frac{f_4^2}{3!} r^{i-1} ,$$

and the series $\sum_{i=1}^{\infty} g_i p^{2i}$ becomes

$$\frac{f_4^2}{6} \sum_{i=1}^{\infty} \frac{1}{16^i} r^{i-1} p^{2i} = \frac{p^2 f_4^2}{16 \cdot 6} \sum_{i=0}^{\infty} \frac{1}{16^i} r^i p^{2i} = \frac{f_4^2/6 + p^2}{16 - r p^2} .$$

We use this expression in (9) and get:

$$T_2(p) = \frac{16\,\alpha_0 - \alpha_0\, r\, p^2 + (f_4^2/6)\, p^2}{16 - r\, p^2 - 16\,\alpha_0\, p^2 + \alpha_0\, r\, p^4 - (f_4^2/6)\, p^4} .$$

Equ. (11) is the equation for the determination of the poles of $T_2(p)$ in the used approximation.

Appendix C: Expansion around another point in complex p^2-plane

We wish to get the propagator for all values of p^2, but our method gives us only series which are expansions around the point $p^2 = 0$. It would be much better to have an expansion of T_2 about any other point of the p^2-plane, say about the point m^2:

$$T_2(p) = \sum_k \alpha_k(m)\,(-p^2 + m^2)^k . \qquad \text{(C 1)}$$

This could be done by writing instead of

$$L = \tfrac{1}{2}\varphi \Box \varphi - \lambda \varphi^4$$

$$L = \tfrac{1}{2}\varphi(\Box + m^2)\,\varphi - \lambda\varphi^4 - \tfrac{1}{2} m^2 \varphi^2 ,$$

considering the last two terms of L as the interaction terms. Our formalism would work in that case as well. The numerical values of the quantities M_k (determining the f_ν) however would not be given by (11) of [1] but by

$$M_k = \int_0^{\infty} y^{2k} \exp\{-i\varepsilon(\lambda y^4 + \tfrac{1}{2} m^2 y^2)\}\, \mathrm{d}y , \qquad \text{(C 2)}$$

$$M_k \cdot M_0 = (2\, i\, \varepsilon\, \lambda)^{-(2k+1)/2}\ \Gamma\left(\frac{2k+1}{2}\right) \cdot \exp\left(-\frac{\varepsilon\, m^4}{32\, i\, \lambda}\right) D_{(2k+1)/2}\left(-\frac{\sqrt{i\,\varepsilon}\, m^2}{2\sqrt{2\,\lambda}}\right) .$$

where $D_n(z)$ is the function of the parabolic cylinder for the index n. The mass renormalization possible in the case $M = 4$, $N = 4$ affects here m in the same manner as p^2.

Further there are some alterations in the calculations resulting from expressions like $(\Box + m^2)\,\delta(0)$ instead of $\Box\,\delta(0)$ and so on. Summing up the new series would be possible now as well, yielding:

$$T_2(p) = \frac{i}{\left(\alpha_0(m) + \sum_{i=1}^{\infty} g_i(m)\,(p^2 - m^2)^i\right)^{-1} - p^2 + m^2} .$$

Apparently, we have

$$\lim_{p^2 \to m^2} T_2(p) = i\,\alpha_0(m) .$$

So it depends only on $\alpha_0(m)$ wheather the point $p^2 = m^2$ is a singular point of the propagator or not. This is very promising as we do not need in such an approach the g_i. As there are no poles in the individual terms of the power series of $\alpha_0(m)$ (with respect to powers of η), we must look for those singularities of $\alpha_0(m)$ which result from the divergence of this series.